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Countable-compact-covering maps and compact-covering maps

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Abstract

We show that every countable-compact-covering map with compact fibers from a separable metric space X onto a first-countable, regular space Y is compact-covering. We also show that the assumption that f is countable-compact-covering cannot be replaced by a weaker condition.

Key words: Compact-covering maps; Countable-compact-covering maps

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0. Introduction

In [10], Michael initiated a systematic study of tri-quotient maps, a concept isolated by Eric K. van Douwen. Prominent examples of tri-quotient maps are open maps, perfect maps, and countable-compact-covering maps with separable fibers defined on metrizable domains.

Tri-quotient maps inherit many preservation properties common to both open and perfect maps (see [6,10]).

There is one possible exception to this pattern: If f is an open or perfect map from a metric space X onto a paracompact space Y with every fiber complete (in the given metric on X), then f is compact-covering. It is not known whether the above remains true when “open or perfect” is replaced by “tri-quotient” (see [10, Question 1.9]).

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Since countable-compact-covering maps with metric domains and separable fibers are examples of tri-quotient maps, the following question, posed as Question 1.1(a) in [13], is a special case that seems to include much of the technical difficulties involved in the more general problem.

Question 0.1. [13, Question 1.1(a)]. Let $f: X \rightarrow Y$ be a map from a separable metrizable space X onto a metrizable space Y , with each $f^{-1}\{y\}$ compact. If f is countable-compact-covering, must f be compact-covering?

In this paper, we give a positive answer to Question 0.1 by proving the following:

Theorem 0.2. *Every countable-compact-covering map $f: X \rightarrow Y$ from a metrizable space X onto a first-countable regular space Y , all whose compact subspaces are separable, with each fiber $f^{-1}\{y\}$ compact, is compact-covering.*

Theorem 0.2 has already lead to some further progress on Question 1.9 of [10] (see [7,8] for details).

Before we can state the other result of the present paper, we need some definitions.

Definition 0.3. Let X be a topological space and let α be an ordinal. The α th derivative of X , denoted by $D^{(\alpha)}X$, is defined inductively as follows:

$$D^{(0)}X = X,$$

$$D^{(\alpha+1)}X = D^{(\alpha)}X \setminus \{x: x \text{ is an isolated point in } D^{(\alpha)}X\},$$

$$D^{(\alpha)}X = \bigcap_{\beta < \alpha} D^{(\beta)}X \text{ for limit ordinals } \alpha.$$

The smallest α for which $D^{(\alpha)}X = D^{(\alpha+1)}X$ is called the *Cantor–Bendixson height* of X (abbreviated CB-height in the sequel) and denoted by $\text{CB}(X)$.

We use the word “map” and the notation $f: X \rightarrow Y$ to designate a continuous surjection.

A map $f: X \rightarrow Y$ is *compact-covering* (respectively *countable-compact-covering*) if every compact (respectively countable, compact) subset of Y is the image of some compact subset of X .

For $\alpha \in \omega_1$, a map $f: X \rightarrow Y$ for a space X onto a space Y is α -compact-covering iff for every countable and compact $E \subset Y$ such that $D^{(\alpha)}E = \emptyset$ there is a compact $C \subset X$ such that $f[C] = E$.

Remark 0.4. A map f is countable-compact-covering if and only if it is α -compact-covering for every $\alpha \in \omega_1$.

Example 0.5. For every $\alpha \in \omega_1$, there is a map $f: X \rightarrow Y$ from a metric space X onto a countable and compact metric space Y with each fiber $f^{-1}\{y\}$ compact that is α -compact-covering, but not $\alpha + 1$ -compact-covering.

In Section 2 of this paper, we prove Theorem 0.2. In Section 3, we construct Example 0.5.

This paper contains the main results of the first author's Ph.D. Thesis [1], as well as the second author's notes [4,5]. The authors would like to thank Gary Gruenhage for pointing out how an earlier partial solution (for zero-dimensional Y) could be generalized to an arbitrary metric Y , as well as Ernest Michael and Howard Wicke for catching many errors in earlier write-ups of these results.

1. Preliminaries

Our terminology is fairly standard. As far as topological concepts are concerned, we follow [2]; set-theoretic notions are mostly used like in [9]. There are a few idiosyncrasies though: $f[A]$ denotes the image of a set A under a function f , the given metric on a metric space X is denoted by μ_X , and \subset stands for what many authors would denote by \subseteq . We shall frequently consider subspaces of a product $X \times Y$. In these situations, π always denotes the projection on the *second* coordinate, i.e., $\pi(\langle x, y \rangle) = y$. Also, for $A(B, C, \dots) \subset X \times Y$ and $y \in Y$, by $A_y(B_y, C_y, \dots)$ we denote the sets $\{x \in X: \langle x, y \rangle \in A(B, C, \dots)\}$. For A , X , Y as above and $E \subset Y$, we call a subset B of A a *lifting of E* iff $\pi[B] = E$.

Now let us mention a few basic tools that we are going to use throughout the paper.

Definition 1.1. Let (X, μ_X) be a fixed metric space. By \mathcal{H} we denote the family of all nonempty compact subsets of X . On \mathcal{H}^2 , we define two functions:

$$d(P, Q) = \inf\{\varepsilon > 0: (\forall q \in Q)(\exists p \in P)[\mu_X(p, q) < \varepsilon]\},$$

$$\rho(P, Q) = \max\{d(P, Q), d(Q, P)\}.$$

The function ρ is a metric (sometimes called the *Hausdorff metric*) which induces a Hausdorff topology on \mathcal{H} , which we shall refer to as the *hyperspace topology*. The hyperspace of a compact space is compact.

The function d is not a metric. However, the family $\{B_\delta(P): \delta > 0, P \in \mathcal{H}\}$ (where $B_\delta(P) = \{Q \in \mathcal{H}: d(P, Q) < \delta\}$) is a base for a topology. We shall refer to this topology as the *d-topology*.

Fact 1.2. *The hyperspace topology is richer than the d-topology. Therefore the d-topology is compact and hereditarily Lindelöf. However it is not Hausdorff; in fact, it is T_0 but not T_1 .*

Also, suppose $P, Q, R, S \in \mathcal{X}$ are such that $P \subset Q$. Then

- (a) $d(P, R) \geq d(Q, R)$,
- (b) $d(R, P) \leq d(R, Q)$,
- (c) $d(R, S) = 0$ iff $S \subset R$,
- (d) $d(Q, S) \leq d(Q, R) + d(R, S)$.

To familiarize ourselves with the concepts just introduced and for later reference, let us consider the following claim.

Claim 1.3. *Let X and Y be as in Theorem 0.2. Suppose $C \subset X \times Y$ is compact. Let $y \in \pi[C]$. Then for every $\delta > 0$ there exists a neighborhood U_δ of y such that $d(C_y, C_{y'}) < \delta$ for all $y' \in U_\delta$.*

We leave the proof of the above claim to the reader. Note that the order of the arguments of the function d in the above claim cannot be reversed.

Finally, let us observe the following.

Remark 1.4. Assume Y is regular and is the continuous image of a separable metric space. Since the continuous image of a Lindelöf space is Lindelöf, it follows that Y is hereditarily Lindelöf. Since a regular Lindelöf space is hereditarily Lindelöf iff it is perfectly normal, we conclude that every closed subspace of Y is a G_δ -subset of Y .

2. Proof of Theorem 0.2

Most of this section will be devoted to the proof of the following.

Theorem 2.1. *Let X be a compact metric space, K a first-countable compact Hausdorff space, and $A \subset X \times K$ be such that every horizontal section of A is nonempty and compact. If the projection $\pi: A \rightarrow K$ onto the second coordinate is countable-compact-covering, then it is compact-covering.*

First we show how Theorem 0.2 follows from Theorem 2.1. To begin with, note that since every separable metric space X can be embedded into a compact metric space, and compactness of subspaces of X is not affected by this embedding, the truth value of Theorem 2.1 does not change if we replace “Let X be a compact metric space” by “Let X be a separable metric space”. Next note that a map $f: X \rightarrow Y$ is compact-covering iff for every compact $K \subset Y$ the restriction of f to $f^{-1}K$ is. Moreover, a map $f: X \rightarrow Y$ is (countable-)compact-covering and has compact fibers iff its graph $\Gamma_f = \{\langle x, y \rangle: f(x) = y\}$ has compact horizontal sections and the projection $\pi: \Gamma_f \rightarrow Y$ of the graph of f onto the second coordinate is (countable-)compact-covering.

Thus Theorem 2.1 implies the following:

Theorem 2.2. *Let $f: X \rightarrow Y$ be a map from a separable metric space X onto a first-countable Hausdorff space Y with every fibre compact. If f is countable-compact-covering, then f is compact-covering.*

Theorem 2.2 differs from Theorem 0.2 only by the additional assumption of separability of X . As we have seen above, it suffices to prove Theorem 2.2 for compact ranges Y . The equivalence of Theorems 0.2 and 2.2 is now implied by the fact that countable-compact-covering maps whose fibres are first-countable Lindelöf spaces are tri-quotient (see [12, 1.1(c)]), [12, Theorem 1.4] which asserts that tri-quotient maps $f: X \rightarrow Y$ with metric domains, regular, countable ranges and completely metrizable fibers are inductively perfect (i.e., there is $X' \subset X$ such that $f[X'] = Y$ and $f|X'$ is perfect), the trivial fact that inductively perfect maps are compact-covering, and the following.

Theorem 2.3. *Let $f: X \rightarrow Y$ be a tri-quotient map from a topological space X onto a topological space Y such that every fiber $f^{-1}\{y\}$ is a separable subspace of X . If Y' is a closed separable subspace of Y , then there is a closed separable $X' \subset X$ such that $f[X'] = Y'$, and the restriction $f|X'$ of f to X' is also tri-quotient.*

Proof. Let us first remind ourselves of the definition of a tri-quotient map.

Definition 2.4. A map $f: X \rightarrow Y$ is *tri-quotient* if there is a t -assignment $U \mapsto U^*$ that assigns an open $U^* \subset Y$ to every open $U \subset X$ such that:

- (a) $U^* \subset f[U]$,
- (b) $X^* = Y$,
- (c) $U \subset V$ implies $U^* \subset V^*$,
- (d) if $y \in U^*$ and \mathcal{W} is a cover of $f^{-1}\{y\} \cap U$ by open subsets of X , then there is a finite $\mathcal{F} \subset \mathcal{W}$ such that $y \in (\bigcup \mathcal{F})^*$.

Proof of Theorem 2.3 (continued). Let f, X, Y, Y' be as in the assumptions. Fix a t -assignment $U \mapsto U^*$ for f . Let $D \subset Y'$ be a countable dense subset of Y' . Define $X' = \text{cl}_X(f^{-1}D)$. Since $f^{-1}D$ is a countable union of separable subspaces of X , it is clear that X' is separable.

Claim 2.5. *If $y \in Y'$ and U is an open subset of X such that $f^{-1}\{y\} \cap X' \cap U = \emptyset$, then $y \notin U^*$.*

Proof. Suppose otherwise, and let y, U be witnesses. Since X' is closed, there is an open cover \mathcal{W} of $U \cap f^{-1}\{y\}$ such that $W \cap X' = \emptyset$ for all $W \in \mathcal{W}$. Let \mathcal{F} be a finite subset of \mathcal{W} such that $y \in F^*$, where F denotes $\bigcup \mathcal{F}$. Since F^* is open, and D is dense in Y' , there is some $z \in D \cap F^*$. However, by the choice of \mathcal{W} , $F \cap f^{-1}D = \emptyset$, which contradicts point (a) of Definition 2.4. \square

Corollary 2.6. $f[X'] = Y'$.

Proof. Since $\text{cl}_Y(Y') = Y'$, it follows from continuity of f that $f[X'] \subset Y'$. For the other inclusion, suppose $y \in Y' \setminus f[X']$. Let $U = X \setminus X'$. Then $f^{-1}\{y\} \subset U$, and thus $y \in U^*$.

This contradicts Claim 2.5 and thus proves the corollary. \square

Proof of Theorem 2.3 (continued). It remains to find a t -assignment for $f|X'$. For $U \subset X'$ that is open in the subspace topology define:

$$U^+ = U \cup (X \setminus X') \quad \text{and} \quad U^\bullet = (U^+)^* \cap Y'.$$

Claim 2.7. *The function $U \mapsto U^\bullet$ is a t -assignment for the function $f|X'$.*

Proof. Clearly, U^\bullet is open in Y' for every U . Also, (b), (c) and (d) of Definition 2.4 are obviously satisfied (for (d), given an open cover \mathcal{W} of $f^{-1}\{y\} \cap U$ associate with it the open cover $\mathcal{W}^+ = \{W^+ : W \in \mathcal{W}\}$).

The only delicate point is (a). So suppose (a) fails, and let $U \subset X'$ and $y \in Y'$ be such that $y \in U^\bullet \setminus f[U]$. Since $f^{-1}\{y\} \cap U = \emptyset$, we have $f^{-1}\{y\} \cap U^+ = f^{-1}\{y\} \cap (X \setminus X')$. Thus, $\{X \setminus X'\}$ is an open cover of $f^{-1}\{y\} \cap U^+$, and since $y \in (U^+)^*$, we must have $y \in (X \setminus X')^*$ by Definition 2.4(d). However, the latter contradicts Claim 2.5. \square

This concludes the proof of Theorem 2.3. \square

The remainder of this section will be devoted to the proof of Theorem 2.1, and most of it in fact to the proof of the key lemma (Lemma 2.9). However, let us first formulate a conjecture that arises in connection with an attempt to solve the still open Question 1.9 of [10]. The key lemma is a special case of this conjecture, one we actually can prove.

Conjecture 2.8. *Let X be a complete separable metric space, K a compact Hausdorff space, and let $A \subset X \times K$ with every horizontal section closed and nonempty. Assume that the projection $\pi : A \rightarrow K$ onto the second coordinate is countable-compact-covering, and let $\delta > 0$. Then there exists a set $C \subset X \times K$ such that:*

- (i) *C is closed and there exists a finite cover \mathcal{W} of C such that every $W \in \mathcal{W}$ is of the form $W = U(W) \times V(W)$, where the diameter of $U(W)$ is less than δ ,*
- (ii) *for all $\langle x, y \rangle \in C$, there exists $x' \in X$ such that $\langle x', y \rangle \in A$ and $\mu_X(x, x') \leq \delta$,*
- (iii) *every countable and compact set $E \subset K$ has a compact lifting $B \subset A \cap C$, i.e., $\pi|A \cap C$ is countable-compact-covering.*

If X is compact rather than must complete, (i)' can be strengthened, and we get the following statement that will be our key lemma.

Lemma 2.9. *Let X be a compact metric space, K a compact Hausdorff space, and let $A \subset X \times K$ with every horizontal section closed and nonempty. Assuming that the projection $\pi : A \rightarrow K$ onto the second coordinate is countable-compact-covering, and let $\delta > 0$. Then there exists a set $C \subset X \times K$ such that:*

- (i) *C is compact,*
- (ii) *for all $\langle x, y \rangle \in C$, there exists $x' \in X$ such that $\langle x', y \rangle \in A$ and $\mu_X(x, x') \leq \delta$,*

(iii) every countable and compact set $E \subset K$ has a compact lifting $B \subset A \cap C$, i.e., $\pi|_{A \cap C}$ is countable-compact-covering.

A set $C \subset X \times K$ that satisfies (i), (ii), and (iii) will be called a δ -approximation of C , where C stands for the lifting of K that we ultimately want to construct.

Now we show how Conjecture 2.8 implies the modification Conjecture 2.1' of Theorem 2.1 that we give below. This argument also shows how Lemma 2.9 implies Theorem 2.1.

Conjecture 2.1'. *Let X be a separable complete metric space, K a first-countable compact Hausdorff space, and $A \subset X \times K$ be such that every horizontal section of A is nonempty and closed. If the projection $\pi: A \rightarrow K$ onto the second coordinate is countable-compact-covering, then it is compact-covering.*

Proof of Conjecture 2.1' from Conjecture 2.8 and of Theorem 2.1 from Lemma 2.9.

Let X, A, K be as in the assumptions of Conjecture 2.1' (respectively Theorem 2.1). We choose a decreasing sequence $(\delta_n)_{n \in \omega}$ of positive reals converging to zero and construct inductively a sequence $(C_n)_{n \in \omega}$ of subsets of $X \times K$, where C_{n+1} is as in Conjecture 2.8 (respectively Lemma 2.9) applied to δ_n in the rôle of δ and A_n in the rôle of A . Define $C_0 = X \times K$, $A_0 = A$, and $A_{n+1} = C_{n+1} \cap A_n$.

Let $C = \bigcap \{A_n; n \in \omega\}$. Since C is by definition a subset of A , the following claim is all that is needed to conclude the proof of Conjecture 2.1' (respectively Theorem 2.1).

Claim 2.10. (a) *The projection of C on the second coordinate is equal to K , i.e., $\pi[C] = K$.*

(b) *C is a compact subset of $X \times K$.*

Proof. For the proof of (a), note that $\pi[C] \subset K$. To show $K \subset \pi[C]$, let $y \in K$. By the construction, we notice that $A_{n+1} \subset A_n$ for every $n \in \omega$ and if $A_{n,y}$ denotes the horizontal sections of A_n at y , then $A_{n+1,y} \subset A_{n,y}$.

So $(A_{n,y})_{n \in \omega}$ is a nested sequence of nonempty closed sets, and each $A_{n,y}$ has a finite cover consisting of sets of diameter less than δ_n each (is actually compact, if X is). Now a standard argument involving König's Tree Lemma shows that $\bigcap_{n \in \omega} A_{n,y} \neq \emptyset$ (in the compact case, the latter is just an intersection of a nested sequence of nonempty compact sets). It follows that $y \in \pi[C]$, which proves (a).

For the proof of (b), it suffices to show that $C = \bigcap \{C_n; n \in \omega\}$. The latter intersection is an intersection of a (not necessarily decreasing) sequence of closed sets, hence closed. And by (i)' and since $\lim_{n \rightarrow \infty} \delta_n = 0$, if K is metric, then C is a totally bounded subset of the complete metric space $X \times K$. If K is not metric, then C is still compact; the proof of the latter is left to the reader. \square

Since $A_n \subset C_n$ for all $n \in \omega$, we have $C \subset \bigcap \{C_n; n \in \omega\}$. To show the other inclusion, let $\langle x, y \rangle \in \bigcap \{C_n; n \in \omega\}$. By Conjecture 2.8 (ii) and the construction, for every $n \in \omega$, there exists $x_n \in X$ such that $\langle x_n, y \rangle \in A_n$ and $\mu_X(x, x_n) \leq \delta_n$. It follows that the sequence $(x_n)_{n \in \omega}$ converges to x in X .

Now, by way of contradiction, if $\langle x, y \rangle \notin C$, then there exist $m \in \omega$ and $\gamma > 0$ such that $B(x, \gamma) \cap A_{m,y} = \emptyset$ (where $B(x, \gamma)$ is the open ball with center x and radius γ).

We find $n > m$ such that $\delta_n < \gamma$. But then $\text{cl}(B(x, \delta_n)) \cap A_{n,y} = \emptyset$, which is impossible, since $x_n \in \text{cl}(B(x, \delta_n)) \cap A_{n,y}$.

This completes the proof of Theorem 2.1 and thus the proof of Theorem 0.2 modulo the proof of Lemma 2.9, which is the subject of the remainder of this section. \square

Proof of Lemma 2.9. For $y \in K$, let $\mathcal{H}(y) = \{P \in \mathcal{H} : P \subset A_y\}$ and $\mathcal{H}_y = \{P \in \mathcal{H}(y) : \text{for every countable and compact } E \subset K \text{ there exists a compact set } C \subset A \text{ such that } \pi[C] = E \text{ and } C_y \subset P\}$.

By $\mathcal{H}_y(A')$ we denote the relativised version of \mathcal{H}_y for $A' \subset A$. Clearly, if $A^s \subset A'$, then $\mathcal{H}_y(A^s) \subset \mathcal{H}_y(A')$.

To state the properties of the families \mathcal{H}_y that are crucial for our proof, we need some more terminology. Let $P \in \mathcal{H}$ and $\delta > 0$. We denote:

$$\text{Fat}(P, \delta) = \{q \in X : d(P, \{q\}) \leq \delta\}.$$

2.11. Crucial Property A. For every $y \in K$, $P \in \mathcal{H}_y$, and $\delta > 0$, there exists a neighborhood U of y such that

(A) for all $z \in U$, there exists $Q \in \mathcal{H}_z$ such that $d(P, Q) < \delta$.

Proof. See [7, Theorem 3.1]. \square

We shall also need a slightly stronger version of Crucial Property A.

2.12. Crucial Property A⁺. For all $y \in K$ and for all $\delta > 0$, there exists a neighborhood U of y such that for every $P \in \mathcal{H}_y$,

(A) for all $z \in U$, there exists $Q \in \mathcal{H}_z$ such that $d(P, Q) < \delta$.

Proof. Since the metric ρ on \mathcal{H}_y is totally bounded, there exists a finite number L such that $\{P_l \in \mathcal{H}_y : l < L\}$ is a $\delta/2$ -dense set. By Crucial Property A, for every $P_l \in \mathcal{H}$, we find a neighborhood U_l of y such that for all $z \in U_l$, there exists a $Q \in \mathcal{H}_z$ with $d(P_l, Q) < \delta/2$.

Let $U = U_0 \cap \dots \cap U_{L-1}$. Then U is a neighborhood of y . We claim that this U works. Let $P \in \mathcal{H}_y$. Then there exists $l < L$ such that $\rho(P, P_l) < \delta/2$. If $z \in U$, then $z \in U_l$ for every $l < L$. So there is some $Q \in \mathcal{H}_z$ such that $d(P_l, Q) < \delta/2$. It follows from Fact 1.2(d) that $d(P, Q) \leq d(P, P_l) + d(P_l, Q) < \delta/2 + \delta/2 = \delta$. Hence $d(P, Q) < \delta$, as desired. \square

2.13. Crucial Property B. Let $y \in K$, $P \in \mathcal{H}_y$, $\delta > 0$, and let U be a neighborhood of y in K . Then (A) implies (B):

(A) for all $z \in U$, there exists $Q \in \mathcal{H}_z$ such that $d(P, Q) < \delta$.

(B) $\pi|_{A \cap (\text{int}(\text{Fat}(P, \delta)) \times U)}$ is countable-compact-covering.

Proof. Let y , P , δ , and U be as in the assumptions. Assume (A) is true. Let $E \subset U$ be countable and compact, and $z \in E$. Then, by Crucial Property A, there exists $Q \in \mathcal{H}_z$ such that $d(P, Q) < \delta$. Let $\gamma = \delta - d(P, Q) > 0$. Now by the definition of \mathcal{H}_z , there exists a compact lifting $D(z) \subset A$ of E such that $(D(z))_z \subset Q$. Since $D(z)$ is compact, by Claim 1.3 there exists a closed neighborhood $O(z)$ of z such that $(D(z))_w \in B_\gamma(Q)$, and hence $d(P, (D(z))_w) \leq d(P, Q) + d(Q, (D(z))_w) < (\delta - \gamma) + \gamma = \delta$ for all $w \in O(z) \cap E$. Thus $D(z) \cap (X \times O(z)) \subset A \cap (\text{int}(\text{Fat}(P, \delta)) \times U)$.

Since E is compact, there is a finite subset $\{z_0, z_1, \dots, z_k\}$ of E such that $\bigcup \{O(z_i) : i \leq k\} \supset E$. Let $D = \bigcup \{D(z_i) \cap (X \times O(z_i)) : i \leq k\}$. Then D is compact, and $D \subset A \cap (\text{int}(\text{Fat}(P, \delta)) \times U)$ and $\pi[D] = E$. \square

The construction of the δ -approximation proceeds in two stages. First we construct a tree T , and then we use induction over T to define the δ -approximation.

Let $\delta > 0$ and $A \subset X \times K$ be as in the assumption of Lemma 2.9. A node t of T will consist of a septuple $(U^t, y^t, M^t, A^t, \sigma^t, P^t, F^t)$, where:

- (1) U^t is a closed subset of K ,
- (2) $y^t \in U^t$,
- (3) M^t is a closed subset of X ,
- (4) $A^t = A \cap (M^t \times U^t)$,
- (5) σ^t is such that $0 < 4\sigma^t < \delta$ and for all $P \in \mathcal{H}_{y^t}(A^t)$ and for all $z \in U^t$, there exists $Q \in \mathcal{H}_z(A^t)$ such that $d(P, Q) < \sigma^t$,
- (6) $P^t = A_{y^t}^t \in \mathcal{H}_{y^t}(A^t)$,
- (7) $F^t = \{z \in U^t : \rho(P^t, A_z^t \cap \text{Fat}(P^t, \sigma^t)) \leq \delta/2\}$,
- (8) for all $z \in U^t \setminus F^t$, there exists a neighborhood $V(z)$ of z in U^t such that $V(z) \cap F^t = \emptyset$ (and thus F^t is a closed subset of U^t), and there exists $\gamma(z) > 0$ such that $2\gamma(z) < \sigma^t$ and $\rho(P^t, R) > \delta/2$ for all $R \in \mathcal{P}(\text{Fat}(A_z^t \cap \text{Fat}(P^t, \sigma^t), 2\gamma(z))) \cap \mathcal{H}$, and moreover, $\text{Fat}(A_z^t \cap \text{Fat}(P^t, \sigma^t), 2\gamma(z)) \subset M^t \subset \text{Fat}(P^t, \delta/2)$.

Note that by Crucial Property B, it follows from (5) and (6) that

- (9) $\pi \upharpoonright_{(A^t \cap (\text{int}(\text{Fat}(P^t, \sigma^t)) \cap M^t) \times U^t)}$ is countable-compact-covering.

For every $t \in T$ and $z \in U^t$, we shall denote $B_z^t = A_z^t \cap \text{Fat}(P^t, \sigma^t)$.

To get started, let $\sigma > 0$ with $4\sigma < \delta$. Then by Crucial Property A⁺ for every $y \in K$, we can choose a closed neighborhood $U(y)$ of y such that for all $P \in \mathcal{H}_y$ and for all $z \in U(y)$, there exists $Q \in \mathcal{H}_z$ such that $d(P, Q) < \sigma$. Since K is compact, there exists a finite set $\{y_i : i \leq n\} \subset K$ such that $\bigcup \{U(y_i) : i \leq n\} \supset K$.

Our tree T will have a finite set of nodes $\{0_i : i \leq n\}$ on its lowest level.

For $i \leq n$, let $U^{0_i} = U(y_i)$, $y^{0_i} = y_i$, $M^{0_i} = X$, $A^{0_i} = A \cap (X \times U^{0_i})$, $\sigma^{0_i} = \sigma$, $P^{0_i} = A_{y^{0_i}}^{0_i}$, and $F^{0_i} = \{z \in U^{0_i} : \rho(P^{0_i}, B_z^{0_i}) \leq \delta/2\}$.

Now suppose a node t has been put into T .

We choose a decreasing sequence $\{G_n^t\}_{n \in \omega}$ of open sets in U^t such that $F^t = \bigcap \{G_n^t : n \in \omega\}$.

Then for every $z \in U^t \setminus F^t$ and for each $n \in \omega$, having chosen the G_n^t , choose $V(z)$, $\gamma(z)$ such that:

(10) $V(z)$ is an open neighborhood of z in U^t and $\gamma(z)$ is a number not exceeding $\min\{\sigma^t/2, 1/(n(z)+1)\}$, where $n(z)$ is the largest $n \in \omega$ such that $z \in G_n^t$, such that $\rho(R, P^t) > \delta/2$ for all $R \in \mathcal{P}(\text{Fat}(B_z^t, 2\gamma(z))) \cap \mathcal{H}$, and moreover

(11) for all $Q \in \mathcal{H}_z \cap \mathcal{P}(B_z^t)$ and for all $w \in V(z)$, there exists $R \in \mathcal{H}_w(A^t)$ such that $d(Q, R) < \gamma(z)$.

Note that $V(z)$ and $\gamma(z)$ that satisfy (10) can be chosen by (8). By Crucial Property A^+ , these objects can be chosen in such a way that (11) also holds.

We define the immediate successors s of t in T . If $F^t = U^t$, then the node t has no successor in T . Otherwise, using the observation made in Remark 1.4, choose a set $\{z_i: i \in \omega\}$ such that $U^t \setminus F^t$ is covered by $\bigcup \{V(z_i): i \in \omega\}$ and

(12) for every n , the set $\{i: V(z_i) \setminus G_n^t \neq \emptyset\}$ is finite.

Let the immediate successors of t be the septuples

$$(U^{s_i}, y^{s_i}, M^{s_i}, A^{s_i}, \sigma^{s_i}, P^{s_i}, F^{s_i})$$

such that for $i \in \omega$,

$$U^{s_i} = U(z_i),$$

$$y^{s_i} = z_i,$$

$$M^{s_i} = \text{Fat}(B_{z_i}^t, 2\gamma(z_i)) \cap M^t,$$

$$A^{s_i} = A^t \cap (M^{s_i} \times U^{s_i}),$$

$$\sigma^{s_i} = \gamma(z_i),$$

$$P^{s_i} = A_{z_i}^{s_i},$$

$$F^{s_i} = \{z \in U^{s_i}: \rho(P^{s_i}, A_z^{s_i} \cap \text{Fat}(P^{s_i}, \sigma^{s_i})) \leq \delta/2\}.$$

This completes the construction of the tree. We show that it works.

First note that the tree thus constructed will obviously satisfy (1)–(7).

Claim 2.14. *At every node t of the tree thus constructed, (8) is satisfied.*

Proof. Let $t \in T$ and let $z \in U^t \setminus F^t$. It follows immediately from the choice of M^t that if γ is sufficiently small, then $\text{Fat}(A_z^t \cap \text{Fat}(P^t, \sigma^t), 2\gamma) \subset M^t \subset \text{Fat}(P^t, \delta/2)$.

We shall show that the first part of (8) is in fact already implied by (1)–(7). So suppose (1)–(7) are satisfied.

Clearly, $B_z^t \subset \text{Fat}(P^t, \sigma^t) \subset \text{Fat}(P^t, \delta/2)$, and hence $d(P^t, B_z^t) < \delta/2$. By the definition of F^t , we must have $\rho(P^t, B_z^t) > \delta/2$. This implies that $d(B_z^t, P^t) > \delta/2$. Now let $\gamma > 0$ be such that $\sigma^t > \gamma$ and

$$(13) \quad d(P^t, B_z^t) + \gamma < \delta/2 \text{ and}$$

$$(14) \quad d(B_z^t, P^t) > \delta/2 + \gamma.$$

By Crucial Properties A and B and since B'_z must contain some $Q \in \mathcal{H}_z(A')$, we find a neighborhood $V \subset U$ of z such that $\pi|_{A' \cap (\text{int}(\text{Fat}(B'_z, \gamma)) \times V)}$ is countable-compact-covering (with respect to subsets of V of course). This is what we want:

By (13) and Fact 1.2 (d), $\text{Fat}(B'_z, \gamma) \subset \text{Fat}(P', \delta/2)$.

Moreover, if $R \in \mathcal{P}(\text{Fat}(B'_z, \gamma)) \cap \mathcal{H}$, i.e., if $d(B'_z, R) \leq \gamma$, then we have $d(B'_z, P') \leq d(B'_z, R) + d(R, P')$, hence by (14)

$$\delta/2 + \gamma - \gamma < d(B'_z, P') - d(B'_z, R) \leq d(R, P').$$

So it follows that $\rho(R, P') > \delta/2$, as desired. \square

Let $<_T$ be the partial order on T (implicitly defined by saying what the immediate successors of each node are).

Observation 2.15. *There is a natural number $N = N(\delta)$ such that every branch of T has length at most N .*

Proof. Let $t \in T$. By (8) and the construction of T , $\rho(M^r, M^t) > \delta/2$ for every successor r of t .

Now Suppose $t_0 <_T t_1 <_T \dots <_T t_K$. By the above remark, $\rho(M^{t_i}, M^{t_j}) > \delta/2$ for $0 \leq i < j \leq K$.

The metric ρ on \mathcal{H} is totally bounded, and hence there is a $\delta/2$ -dense set of size N in \mathcal{H} for some finite N . It follows that $K < N$. \square

Now recall the reason why a certain node t of T may not have any successors: the only possibility is that $F^t = U^t$. We make another observation.

Observation 2.16. *If $F^t = U^t$, then $M^t \times U^t$ is a δ -approximation of $C|_{U^t}$.*

Proof. We have to establish three properties of $M^t \times U^t$.

For (i), $M^t \times U^t$ is clearly compact.

For (ii), note that $\rho(M^t, B'_z) \leq \delta$ by (7), the fact that $M^t \subset \text{Fat}(P', \delta/2)$, and the triangle inequality for the function d . Now it follows from the definition of d that there is $x' \in B'_z(\subset A'_z)$ with $\mu_X(x, x') \leq \delta$.

Property (iii) $_{U^t}$ follows immediately from (9). \square

Now let us construct, by induction on $t \in T$, δ -approximations C^t of $C|_{U^t}$. For a leaf $t \in T$, let $C^t = M^t \times U^t$.

Let $t \in T$ be a node, let $S(t)$ be the set of all immediate successors of t and suppose C^s has been defined for all $s \in S(t)$. We put $C^t = (M^t \times F^t) \cup \bigcup \{C^s : s \in S(t)\}$.

Let $T_0 = \{t \in T : t \text{ is not an immediate successor of any node in } T\}$. By construction, T_0 is finite.

Now let $C = \bigcup \{C^t : t \in T_0\}$. The following is all we need for the proof of Lemma 2.9.

Sublemma 2.17. *C is a δ -approximation of C .*

Proof. Since T_0 is finite, it suffices to show by induction over T that C^t is a δ -approximation of $C|_{U^t}$.

By Observation 2.16, this is in fact the case for every leave t of T .

Now suppose $t \in T$ is not a leave and for every successor s of t the set C^s is a δ -approximation of $C|_{U^s}$. We have to verify three properties.

(i) C^t is compact.

Since C^t is a subset of the compact space $M^t \times U^t$, it suffices to show that C^t is closed. By (12), $C^t \cap (X \times (U^t \setminus G_n^t))$ is a union of finitely many closed sets, hence closed. In other words, $\text{cl}(C^t) \setminus C^t \subset X \times F^t$. But by the construction of the tree, every horizontal section of C^t is contained in M^t , hence $\text{cl}(C^t) \setminus C^t \subset M^t \times F^t \subset C^t$, which is a fancy way of saying that C^t is closed.

(ii) for all $\langle x, y \rangle \in C^t$, there exists $x' \in X$ such that $\mu_X(x, x') \leq \delta$ and $\langle x', y \rangle \in A$.

Let $\langle x, y \rangle \in C^t$. Consider two cases.

Case 1: $y \in F^t$. Then we find x' as in the proof of Observation 2.16.

Case 2: $y \notin F^t$. Then $\langle x, y \rangle \in C^s$ for some immediate successor s of t , and by the inductive assumption there is some $x' \in X$ such that $\mu_X(x, x') \leq \delta$ and $\langle x', y \rangle \in A$.

It remains to show that $\pi|_{C^t \cap A}$ is countable-compact-covering with respect to subsets of U^t . Let $\tilde{A}^t = C^t \cap A^t$.

To make the inductive argument work, we actually show something slightly stronger.

Claim 2.18. *For every $t \in T$, for every countable and compact set $E \subset U^t$, for every $P \in \mathcal{H}_{y^t}(A^t)$, there exists a compact set $D^t \subset \tilde{A}^t \cap (\text{Fat}(P, 2\sigma^t) \times U^t)$ such that $\pi[D^t] = E$.*

Proof. We prove this by induction on $t \in T$.

Let t be a leaf and $P \in \mathcal{H}_{y^t}(A^t)$. Let $E \subset U^t$ be countable and compact. By (9), there exists a compact lifting $D' \subset \tilde{A}^t \cap (\text{Fat}(P, \sigma^t) \times U^t)$ of E . By the definition of $\mathcal{H}_{y^t}(A^t)$ there exists also a compact lifting $D'' \subset A^t (= \tilde{A}^t)$ such that $D'' \subset P$. By Claim 1.3, we have $D'' \cap (X \times V) \subset \tilde{A}^t \cap (\text{Fat}(P, \sigma^t) \times U^t)$ for some closed neighborhood V of y . Let $D^t = D'' \cap (X \times V) \cup (D' \setminus (X \times \text{int}(V)))$. This D^t is as required. \square

The inductive step boils down to the following.

Claim 2.19. *Let $P \in \mathcal{H}_{y^t}(A^t)$. Let $E \subset U^t$ be countable and compact, and let $\{s_i: i \in \omega\}$ be the set of all immediate successors of t . Denote y^{s_i} by z_i . Suppose for every $i \in \omega$ we are given $\tilde{A}^{s_i} \subset A^{s_i}$ such that for every $Q \in \mathcal{H}_{z_i}(A^{s_i})$ there is a compact $D^i \subset \tilde{A}^{s_i} \cap (\text{Fat}(Q, 2\sigma^{s_i}) \times U^{s_i})$ such that $\pi[D^i] = E \cap U^{s_i}$. Then there is a compact $D \subset A^t \cap (\text{Fat}(P, 2\sigma^t) \times U^t)$ such that $\pi[D] = E$ and for every $i \in \omega$ we have $D \cap (X \times U^{s_i}) \subset \tilde{A}^{s_i}$.*

Proof. First note that by the choice of γ in (10) we have:

(15) If s is a successor of t in T , then $2\sigma^s < \sigma^t$.

Now suppose E , t , and P are as in the assumptions of Claim 2.19. Consider the set $E \cap F^t$. This is countable and compact. We prove the claim by induction over the CB-height of $E \cap F^t$. More precisely, we shall prove the following version of the claim which clearly suffices, since one can partition any countable and compact set E into finitely many countable and compact sets whose highest nonvanishing CB-derivatives are singletons.

Claim 2.20. *Let $P \in \mathcal{H}_{y^t}(A^t)$. Let $E \subset U^t$ be countable and compact such that $D^{(\omega)}(E \cap F^t) = \{y\}$ for some y . Let $P' \in \mathcal{H}_y(A^t)$ be such that $d(P, P') \leq \sigma^t/2$. Suppose for every $i \in \omega$, we are given $\tilde{A}^{s_i} \subset A^{s_i}$ such that for every $Q \in \mathcal{H}_{z_i}(A^{s_i})$ there is a compact set $D^i \subset \tilde{A}^{s_i} \cap (\text{Fat}(Q, 2\sigma^{s_i}) \times U^{s_i})$ such that $\pi[D^i] = E \cap U^{s_i}$. Then there is a compact set $D \subset A^t \cap (\text{Fat}(P, 2\sigma^t) \times U^t)$ such that $\pi[D] = E$ and for every $i \in \omega$, we have $D \cap (X \times U^{s_i}) \subset \tilde{A}^{s_i}$. Moreover, $D \cap (X \times \{y\}) \subset P' \times \{y\}$.*

Proof. Strictly speaking, the first case we have to consider does not fit entirely into the framework of Claim 2.20. We hope the reader will forgive us the sacrifice of formal correctness made in an attempt to keep the statement of the claim only moderately long.

Case 1: $E \cap F^t = \emptyset$.

By (5), for every $i \in \omega$, there exists $Q(z_i) \in \mathcal{H}_{z_i}(A^t)$ such that $Q(z_i) \subset \text{int}(\text{Fat}(P, \sigma^t))$. By the assumption, for every $i \in \omega$ there is a compact lifting D^i of $E \cap U^{s_i}$ such that

$$D^i \subset \tilde{A}^{s_i} \cap (\text{Fat}(Q(z_i), 2\sigma^{s_i}) \times U^{s_i}).$$

By (15) and Fact 1.2 (d), $\text{Fat}(Q(z_i), 2\sigma^{s_i}) \subset \text{Fat}(P, 2\sigma^t)$. Moreover, $A^{s_i} \subset A^t$ for each $i \in \omega$. Therefore, each $D^i \subset A^t \cap (\text{Fat}(P, 2\sigma^t) \times U^t)$.

Since $E \cap F^t = \emptyset$ and $E \subset U^t$ is compact, there exists $n \in \omega$ such that $E \subset U^t \setminus G_n^t$. So by (12), there are only finitely many i such that $E \cap U_i \neq \emptyset$. Hence $D^i = \emptyset$ for all but finitely many i ; therefore the union $\bigcup \{D^i : i \in \omega\}$ is compact and thus as required.

Case 2: $\alpha = 0$ (i.e., $E \cap F^t = \{y\}$ for some y).

Choose a decreasing sequence $(\varepsilon_j)_{j \in \omega}$ of positive reals converging to zero such that $2\varepsilon_0 = \sigma^t - d(P, P')$.

Then, by Crucial Property A and regularity of K , there exists a decreasing base $(W_{\varepsilon_j})_{j \in \omega}$ of closed neighborhoods of y such that for all $z \in W_{\varepsilon_j}$, there exists $Q(z) \in \mathcal{H}_z(A^t)$ such that $d(P', Q(z)) < \varepsilon_j$. Moreover, by shrinking each W_{ε_j} if necessary, we may require that if $U^{s_i} \subset W_{\varepsilon_j}$, then $2\sigma^{s_i} \leq 2/(n(z_i) + 1) < \varepsilon_j$ (see (10) for the definition of $n(z_i)$).

Now for every i such that $U^{s_i} \cap E \neq \emptyset$, choose $Q(z_i)$ such that $d(P', Q(z_i)) < \varepsilon_j$ for each of the finitely many j for which $U^{s_i} \subset W_{\varepsilon_j}$. Using the inductive assumption, choose a compact set $D^i \subset \tilde{A}^{s_i} \cap (\text{Fat}(Q(z_i), 2\sigma^{s_i}) \times U^{s_i})$ such that $\pi[D^i] = E \cap U^{s_i}$.

Subclaim 2.21. *For fixed ε_j and i as above, $d(P', Z) \leq 2\varepsilon_j$ for every horizontal section Z of D^i .*

Proof. First, we let $n(z_i, j) = \max\{n: U^{s_i} \subset G_n^t \cap W_{\varepsilon_j} \text{ and } 2/(n+1) < \varepsilon_j\}$, where G_n^t is as in Case 1 in the proof of Claim 2.20.

Recall that $n(z_i)$ is the largest n such that $z_i \in U^{s_i} \subset G_n^t$. Therefore,

$$n(z_i, j) \leq n(z_i). \quad (*)$$

It follows from $(*)$ and (10) that $d(P', Z) \leq d(P', Q(z_i)) + d(Q(z_i), Z) < \varepsilon_j + 2\sigma^{s_i} = \varepsilon_j + 2\gamma(z_i) \leq \varepsilon_j + 2/(n(z_i) + 1) \leq \varepsilon_j + 2/(n(z_i, j) + 1) \leq \varepsilon_j + \varepsilon_j = 2\varepsilon_j$. (This is why we require $\gamma(z_i) \leq \min\{\sigma^t/2, 2(n(z_i) + 1)\}$ in (10).) \square

Proof of Claim 2.20 (continued). By (12) and the fact $F^t \cap E = \{y\}$, there are only finitely many i such that $U^{s_i} \cap E \neq \emptyset$ and U^{s_i} is not a subset of W_{ε_0} ; say for $i > i_0$, we have $U^{s_i} \subset W_{\varepsilon_0}$. As we have already shown in Case 1, we can choose a compact lifting $D^0 \subset \bigcup_{i \leq i_0} \tilde{A}^{s_i} \cap (\text{Fat}(P, 2\sigma^t) \times U^{s_i})$ of $E^0 = E \cap \bigcup_{i \leq i_0} U^{s_i}$. Next we choose an increasing sequence $(i_k)_{k \in \omega}$ of natural numbers such that for all $i > i_k$, we have $U^{s_i} \subset W_{\varepsilon_k}$. For every i such that $i_k < i \leq i_{k+1}$ we choose a compact lifting $D^i \subset \tilde{A}^{s_i} \cap (\text{Fat}(P', 2\varepsilon_k) \times U^{s_i})$ of $E \cap U^{s_i}$ (such liftings exist by Subclaim 2.21).

Now let $D = \text{cl}(D^0 \cup \bigcup_{i > i_0} D^i)$. It is not hard to see that $\pi[D] = E$, that $D \subset \text{Fat}(P, 2\sigma^t) \times U^t$, that $D \cap (X \times U^{s_i}) \subset \tilde{A}^{s_i}$ for all $i \in \omega$, and that $D \setminus (D^0 \cup \bigcup_{i > i_0} D^i) \subset P' \times \{y\}$. In particular, $D \subset A^t$. So D is the required lifting.

Case 3: Now suppose claim 2.20 holds for all $\beta < \alpha$.

Choose a decreasing sequence $(\varepsilon_k)_{k \in \omega}$ of positive reals convergent to zero such that $\varepsilon_0 \leq \sigma^t$ and a neighborhood base $(V_n)_{n \in \omega}$ of open (in U^t) neighborhoods of y such that $\text{cl}(V_{n+1}) \subset V_n$ for all $n \in \omega$. Denote $E_n = E \cap (V_n \setminus \text{int}(V_{n+1}))$. Each E_n is countable, compact, and $D^{(\alpha)}(E_n \cap F^t) = \emptyset$. For every $z \in E \cap F^t$, choose $Q(z) \in \mathcal{K}_z(A^t)$ such that $d(p, Q(z)) < \sigma^t$ (this is possible by (5)). Moreover, let us choose the $Q(z)$ in such a way that for every k there exists $n(k)$ such that for all $z \in E \cap V_{n(k)}$ we have $d(P', Q(z)) < \varepsilon_k$. This is possible by Crucial Property A.

Note that

(16) for $\varepsilon_k < \sigma^t - d(P, P')$ the inequality $d(P, Q(z)) < \sigma^t$ is already implied by $d(P', Q(z)) < \varepsilon_k$.

For simplicity of notation assume that $n(k) = k$ for all k .

Fact 2.22. For every k and every $z \in (V_k \setminus \text{cl}(V_{k+1})) \cap (E \cap F^t)$ there exist a closed neighborhood $U(z) \subset V_k \setminus \text{cl}(V_{k+1})$ of z and compact lifting $D(z)$ of $E \cap U(z)$ such that

- (a) $D(z) \subset A^t \cap (\text{Fat}(Q(y), \varepsilon_k)) \times U^t$ and
- (b) $D(z) \cap (X \times U^{s_i}) \subset \tilde{A}^{s_i}$ for all $i \in \omega$.

Proof. Fix k and $z \in (V_k \setminus \text{cl}(V_{k+1})) \cap F^t$. First, choose a closed neighborhood $U' \subset V_k \setminus \text{cl}(V_{k+1})$ of z such that the highest nonvanishing CB-derivative $D^{(\beta)}(E \cap F^t \cap U') = \{z\}$ for some z . Necessarily, $\beta < \alpha$ since $D^{(\alpha)}(E \cap F^t \cap U') = \emptyset$ as $D^{(\alpha)}((E \cap F^t) \cap (V_k \setminus V_{k+1})) = \emptyset$. By (16), let $Q(z) \in \mathcal{K}_z(A^t)$ such that $d(P, Q(z)) < \sigma^t$.

Now use the inductive assumption with $Q(z)$ in the rôle of P' (and P playing itself) to get a compact lifting D' of $E \cap U'$ that satisfies (b). D' may not satisfy

(a), but since the horizontal section of D' at z is contained in $Q(z)$ (by making $\epsilon_0 < \sigma' - d(P, P')$), Claim 1.3 implies that there is a closed neighborhood U'' of z such that $U'' \subset U'$ and $D' \cap (X \times U'')$ does satisfy (a). \square

Proof of Claim 2.20 (continued). Now assume for every $z \in V_0 \setminus \{y\}$ we have chosen $U(z)$ and $D(z)$ as in Fact 2.22. Since E_k is compact, for every k there exist $z_0^k, \dots, z_{l(k)}^k$ such that E_k is covered by $\bigcup \{U(z_j^k): j \leq l(k)\}$. Let $D^k = \bigcup \{D(z_j^k): j \leq l(k)\}$. Let $D^- = \bigcup_{k \in \omega} D^k \cup (P' \times \{y\})$. Then D^- is a compact lifting of the set

$$E^- = \left\{ w \in E: (\exists k)(\exists j \leq l(k)) \left[w \in U(y_j^k) \right] \right\} \cup \{y\}$$

and

$$D^- \cap (X \times U^{s_i}) \subset \tilde{A}^{s_i} \quad \text{for all } i \in \omega.$$

In other words, if $E^- = E$, then we are done. However, generally this may not be the case: First of all, notice that the set $E' = (E \cap F') \setminus V_0$ may not be empty. (We cannot arbitrarily require that $V_0 = U'$, since we assumed that $V_0 = V_{n(0)}$, and $\epsilon_0 \leq \sigma'$.) However, $D^{(\alpha)}(E') = \emptyset$, and y is not in the closure of E' , so there isn't much to worry about: let D^* be a compact lifting of $E^* = E \setminus V_0$ such that $D^* \cap (X \times U^{s_i}) \subset \tilde{A}^{s_i}$ for all $i \in \omega$ and $D^* \subset A' \cap (\text{Fat}(P, 2\sigma') \times U')$. Such a D^* exists by the inductive assumption.

Now let $E^+ = (E \setminus V_0) \cup \{y\}$. It is not hard to see that E^+ is countable and compact, and such that $E^+ \cap F' = \{y\}$. We know already from our work in Case 2 how to find a compact lifting D^+ and E^+ such that $D = D^* \cup D^- \cup D^+$ is as required. \square

This concludes the proofs of Theorems 2.1 and 0.2.

3. α -compact-covering does not suffice

In the introduction, we defined the notion of an α -compact-covering map. In the present section, we show that for any fixed countable ordinal α , in Theorem 0.2, the assumption that f is countable-compact-covering cannot be weakened to being α -compact-covering. It was already observed by Michael [11, (3)] that 2-compact-covering (“sequence-covering” in Michael’s terminology) does not suffice and Steprāns and Watson observed [14] that there is no fixed $n \in \omega$ so that the assumption that f is n -compact-covering would suffice for the proof of Theorem 0.2.

We show that for every $\alpha \in \omega_1 \setminus \{0\}$, there exists an α -compact-covering map $f: X \rightarrow Y$ for a metric space X onto a countable and compact metric space Y with each fibre compact that is α -compact-covering, but not compact-covering.

More precisely, we show the following:

Theorem 3.1. *For every $\alpha \in \omega_1 \setminus \{0\}$, for every $y' \in [0, 1]$, and for every open neighborhood $V \subset [0, 1]$ of y' , there exist: a subspace $X_\alpha \subset [0, 1]^2$, each horizontal*

section of which is compact, a countable and compact subspace $E_\alpha \subset V$ with $D^{(\alpha)}E_\alpha = \{y'\}$ such that the projection π onto the second coordinate maps X_α onto E_α and $\pi|_{X_\alpha}$ is α -compact-covering, but E_α itself is not the image of a compact subspace of X_α (hence $\pi|_{X_\alpha}$ is not $\alpha + 1$ -compact-covering).

Proof. We prove this by induction on $\alpha \geq 1$.

Case 1: $\alpha = 1$.

We fix $y' \in [0, 1]$ and an open neighborhood V of y' , and choose a one-to-one sequence $(y_n)_{n \in \omega}$ in V such that $y_n \rightarrow y'$.

Let $E_1 = \{y_n : n \in \omega\} \cap \{y'\}$ and let $X_1 = \{\langle 1, y_n \rangle : n \in \omega\} \cup \{\langle 0, y' \rangle\}$.

Then each horizontal section of X_1 is a singleton, hence compact, and $\pi[X_1] = E_1$. Since “1-compact-covering” is the same as being a surjection, $\pi|_{X_1}$ is 1-compact-covering.

Also, E_1 cannot be lifted to a compact subset of X_1 ; the only candidate for a lifting would be X_1 itself, which is not compact. Since $D^{(2)}E_1 = \emptyset$, the map $\pi|_{X_1}$ is not 2-compact-covering.

Case 2: Suppose $\alpha > 1$ and the statement is true for all $0 < \beta < \alpha$.

We choose: $y' \in [0, 1]$, an open neighborhood V of y' , a one-to-one sequence $(y_n)_{n \in \omega}$ in V such that $y_n \rightarrow y'$, a decreasing neighborhood base $(O_n)_{n \in \omega}$ in V at y' and a sequence $(U_n)_{n \in \omega}$ of open sets such that each $y_n \in U_n \subset (O_n \setminus O_{n+1})$.

Furthermore, choose a sequence $(\alpha_n)_{n \in \omega}$ of ordinals $< \alpha$ such that

- if $\alpha = \beta + 1$ for some β , then $\alpha_n = \beta$ for all n ,
- if α is a limit ordinal, then the sequence $(\alpha_n)_{n \in \omega}$ is increasing and $\lim_{n \rightarrow \infty} \alpha_n = \alpha$.

By the inductive assumption, for each $n \in \omega$, there exist a countable and compact subset E_{α_n} of U_n such that $D^{(\alpha_n)}E_{\alpha_n} = \{y_n\}$, and a subset X_{α_n} of $[0, 1]^2$ such that every horizontal section of X_{α_n} is compact, $\pi[X_{\alpha_n}] = E_{\alpha_n}$, and $\pi|_{X_{\alpha_n}}$ is α_n -compact-covering, but E_{α_n} itself has no compact lifting in X_{α_n} .

We can multiply the first coordinates of points in each X_{α_n} by $\frac{1}{2}$ to obtain a homeomorphic image of X_{α_n} in $[0, \frac{1}{2}] \times U_n$ which has the properties stated for $\pi|_{X_{\alpha_n}}$. For simplicity of notation, we will denote this homeomorphic image by X_{α_n} again.

Let $E_\alpha = (\bigcup_{n \in \omega} E_{\alpha_n}) \cup \{y'\}$.

Then E_α is countable and compact, and moreover, $D^{(\alpha)}E_\alpha = \{y'\}$.

Now let

$$X_\alpha = \left(\bigcup_{n \in \omega} X_{\alpha_n} \right) \cup \left(\bigcup_{n \in \omega} \{1\} \times E_{\alpha_n} \right) \cup \left(\left[0, \frac{1}{2}\right] \times \{y'\} \right).$$

Clearly, every horizontal section of X_α is compact.

We show that $\pi|_{X_\alpha}$ is α -compact-covering.

Let $E \subset E_\alpha$ be countable and compact with $D^{(\alpha)}E = \emptyset$.

Note that since in particular $y' \notin D^{(\alpha)}E$, for some $m \in \omega$ and all $n > m$ we must have $D^{(\alpha_n)}(E \cap E_{\alpha_n}) = \emptyset$, and moreover, if $y' \notin E$, then $(E \cap E_{\alpha_n}) = \emptyset$ for all $n > m$. So by the inductive assumption, for all $n > m$ there are compact $C_n \subset X_{\alpha_n}$ with $\pi[C_n] = E \cap E_{\alpha_n}$.

If $y' \in E$, then put

$$C = \left(\bigcup_{n \leq m} (\{1\} \times (E \cap E_{\alpha_n})) \right) \cup \left(\bigcup_{n > m} C_n \right) \cup \left([0, \frac{1}{2}] \times \{y'\} \right).$$

If $y' \notin E$, put

$$C = \left(\bigcup_{n \leq m} (\{1\} \times (E \cap E_{\alpha_n})) \right).$$

It is now not hard to see that in either case, C is a compact lifting of E .

It remains to show that X_α contains no compact lifting C of E_α . Suppose there were such C . Then $C_n = C \cap ([0, 1] \times E_{\alpha_n})$ is a compact lifting of E_{α_n} for every n , and thus, by the inductive assumption, C_n is not contained in X_{α_n} . In other words, for every n , there is some $y'_n \in E_{\alpha_n}$ with $\langle 1, y'_n \rangle \in C_n \subset C$. Then $\lim_{n \rightarrow \infty} y'_n = y'$ by the choice of the U_n , and thus $\langle 1, y' \rangle \in C$ by compactness of C . The latter is impossible, since $C \subset X_\alpha$ and $\langle 1, y' \rangle \notin X_\alpha$. \square

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